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Central extensions of the quasi-orthogonal Lie algebras

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Abstract. We determine the central extensions of a whole family of Lie algebras, obtained by the method of graded contractions from $so(N+1)$, N arbitrary. All the inhomogeneous orthogonal and pseudo-orthogonal algebras are members of this family, as well as a large number of other non-semisimple algebras, all of which have at least a semidirect structure (in some cases two or more). The dimensions of their second cohomology groups $H^2(\mathcal{G}, \mathbb{R})$ and the explicit expression of their central extensions are given.

1. Introduction

This paper is devoted to investigating the second cohomology groups of a large class of algebras, the so-called orthogonal Cayley–Klein (CK) family of algebras. These algebras may be obtained by a sequence of ordinary contractions starting from $so(N+1)$ or from $so(p,q)$ and can be described by using the alternative method of graded contractions [1, 2].

The problem of finding the cohomology groups for this family of algebras is primarily of mathematical interest but it is not devoid of a physical one since the CK algebras include all kinematical algebras of physical relevance. There are (at least) three main areas where central extensions play a role in physics. First, the existence of a non-trivial cohomology group is associated with projective representations, a fact discussed first in general by Bargmann [3] but which has its roots in the work of Weyl [4] and in the classic paper of Wigner [5]. Second, in the Kirillov–Kostant–Souriau theory, homogeneous symplectic manifolds under a group appear as the orbits of the coadjoint representation of either the group itself or of a central extension. Third, if a group is considered as the invariance group of a given physical theory, the most general Lagrangian which leads to invariant equations of motion is not necessarily a strictly invariant Lagrangian, but a quasi-invariant one; again this is linked to the central extensions of the group [6].

The second cohomology group is trivial for semisimple Lie algebras. For some specific non-semisimple algebras it has also been studied (e.g. for Euclidean, Poincaré and Galilei algebras in N dimensions). It is well known [3, 7] that the Galilei group admits projective representations and the Lie algebra statement $H^2(\mathcal{G}(3+1), \mathbb{R}) = \mathbb{R}$ has its counterpart in the fact that the group can be centrally extended by $U(1)$ (the phase group involved in the ray [3] representations). The cohomology of these algebras in low dimensions is also

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known. However, some recent rediscoveries (as for the $(2+1)$ Galilei algebra) suggest that part of these results have become buried in the literature.

Moving from the semisimple $so(p,q)$ algebras by successive contractions it is found that, as a rule, the cohomology groups of the resulting algebras become larger. However, inhomogeneity is not *per se* a sufficient condition for the existence of non-trivial cohomology, as exhibited by the contrast between the 10-dimensional Galilei and Poincaré algebras. Moreover, as far as we know, there is no systematic study of the second cohomology groups covering a large family of algebras, so that the present study may help to describe the relationship between cohomology and contractions. It should be noted, however, that the dimension of the Lie algebra cohomology groups does not need to be the same as the dimension of the $U(1)$-valued group cohomology. This is because when moving from the Lie algebras to Lie groups the topology of the groups comes into play and this may reduce the dimension of the different cohomology groups (for a general statement see [3] and [8]). In Bargmann's terminology [3] (see [9] for an outlook) it is not always possible to extend a local exponent to the whole group (it is possible if the group is simply connected, hence the role of the universal covering group).

This paper is organized as follows. The algebras in the orthogonal CK family are introduced in section 2, as well as the way the relevant kinematical Lie algebras are included in it. Section 3 gives a reasonably complete account of the procedure used to find the central extensions of the orthogonal CK algebras (including the characterization of trivial and equivalent extensions). This section may be omitted completely if the reader is not interested in the computational details. In section 4 we state the final solution and provide a method to compute the dimension of the second cohomology group for any CK algebra in any dimension. The method is illustrated in section 5, where all central extensions for the lower dimensional CK orthogonal algebras are explicitly described, and some comments on the *group* cohomology for the CK quasi-orthogonal groups are made. Some concluding remarks and prospects for future work close the paper.

2. The CK family of quasi-orthogonal algebras as graded contractions of $so(N+1)$

Let us first introduce the set of algebras we are going to deal with in connection with graded-contraction theory [1, 2], which is a convenient tool for our aims in this paper. The starting point is the real Lie algebra $so(N+1)$ with $N(N+1)/2$ generators Ω_{ab} $(a, b = 0, 1, \dots, N,\ a < b)$. Its non-zero Lie brackets are

$$[\Omega_{ab}, \Omega_{ac}] = \Omega_{bc} \qquad [\Omega_{ab}, \Omega_{bc}] = -\Omega_{ac} \qquad [\Omega_{ac}, \Omega_{bc}] = \Omega_{ab} \qquad a < b < c \tag{2.1}$$

(commutators involving four different indices are zero). This Lie algebra can be endowed with a fine grading group $\mathbb{Z}_2^{\otimes N}$. The corresponding graded contractions of $so(N+1)$ constitute a large set of Lie algebras, depending on $2^N - 1$ real contraction parameters [10], which include from the simple Lie algebras $so(p,q)$ (when all parameters are different from zero) to the Abelian algebra in the opposite case, when all parameters are equal to zero. Properties associated with the simplicity of the algebra are lost at some point beyond the simple algebras in the contraction lattice, yet there exists a particular subset or family of these graded contractions, nearer to the simple ones, which essentially preserve these properties and may therefore be called [11] 'quasi-simple' algebras. This family [12] encompasses the pseudo-orthogonal algebras (the B_l and D_l Cartan series) as well as their nearest non-simple contractions; collectively, all these algebras are called *quasi-orthogonal*.

In this paper we will deal exclusively with this family† to be defined precisely below. Its members are called CK algebras since they are exactly the family of motion algebras of the geometries of a real space with a projective metric in the CK sense [13, 14]. If the graded contraction procedure is implicitly assumed, the algebras in the CK family may be referred to as contractions of the compact $so(N+1)$, although in this sentence the word 'contraction' must be adequately understood and the compactness is lost.

The set of CK Lie algebras depends on N real coefficients $\omega_1, \ldots, \omega_N$ which codify in a convenient way the information on the Lie algebra structure. In terms of

$$\omega_{ab} := \omega_{a+1}\omega_{a+2}\ldots\omega_b \qquad (a, b = 0, 1, \ldots, N,\ a < b) \qquad \omega_{aa} := 1, \tag{2.2}$$

for which we remark the relations

$$\omega_{ac} = \omega_{ab}\omega_{bc} \qquad a \leqslant b \leqslant c \qquad \omega_a = \omega_{a-1\,a} \qquad a = 1, \ldots, N, \tag{2.3}$$

the independent non-vanishing commutation relations in the contracted CK Lie algebra are given (cf (2.1)) by

$$[\Omega_{ab}, \Omega_{ac}] = \omega_{ab}\Omega_{bc} \qquad [\Omega_{ab}, \Omega_{bc}] = -\Omega_{ac} \qquad [\Omega_{ac}, \Omega_{bc}] = \omega_{bc}\Omega_{ab} \qquad a < b < c. \tag{2.4}$$

This CK or quasi-orthogonal algebra will be denoted as $so_{\omega_1,\ldots,\omega_N}(N+1)$, making explicit the parameters ω_i. This has a (vector) representation by $(N+1)\times(N+1)$ real matrices, given by

$$\Omega_{ab} = -\omega_{ab}e_{ab} + e_{ba} \tag{2.5}$$

where e_{ab} is the matrix with a single non-zero entry, 1, in the row a and column b. For $\omega_1 = \cdots = \omega_N = 1$, we recover the compact $so(N+1)$ algebra. Since each coefficient ω_a may take positive, negative or zero values, and by means of a simple rescaling of the initial generators it can be reduced to the standard values of 1, −1 or 0, it is clear that the family $so_{\omega_1,\ldots,\omega_N}(N+1)$ includes 3^N Lie algebras. Some of these can be isomorphic; for instance,

$$so_{\omega_1,\omega_2,\ldots,\omega_{N-1},\omega_N}(N+1) \simeq so_{\omega_N,\omega_{N-1},\ldots,\omega_2,\omega_1}(N+1). \tag{2.6}$$

The family $so_{\omega_1,\ldots,\omega_N}(N+1)$ of CK algebras includes algebras of physical interest [15]. The structure of these algebras can be characterized by two statements.

- When *all* constants $\omega_a \neq 0$, the algebra $so_{\omega_1,\ldots,\omega_N}(N+1)$ is a simple (barring the special $N = 3$ case) real Lie algebra in the Cartan series B_l or D_l, and it is isomorphic to a pseudo-orthogonal algebra $so(p, q)$ with $p + q = N + 1$.
- If a constant $\omega_a = 0$, for $a = 1, \ldots, N$, the resulting algebra $so_{\omega_1,\ldots,\omega_a=0,\ldots,\omega_N}(N+1)$ has the semidirect structure (see equation (2.4))

$$so_{\omega_1,\ldots,\omega_{a-1},\omega_a=0,\omega_{a+1},\ldots,\omega_N}(N+1) \equiv t \odot (so_{\omega_1,\ldots,\omega_{a-1}}(a) \oplus so_{\omega_{a+1},\ldots,\omega_N}(N+1-a)) \tag{2.7}$$

where t is an Abelian subalgebra $\dim t = a(N+1-a)$ and the remaining subalgebra is a direct sum. The three subalgebras appearing in (2.7) are generated by

$$\begin{aligned} t &= \langle \Omega_{ij};\ i = 0, 1, \ldots, a-1;\ \ j = a, a+1, \ldots, N\rangle \\ so_{\omega_1,\ldots,\omega_{a-1}}(a) &= \langle \Omega_{ij};\ i, j = 0, 1, \ldots, a-1\rangle \\ so_{\omega_{a+1},\ldots,\omega_N}(N+1-a) &= \langle \Omega_{ij};\ i, j = a, a+1, \ldots, N\rangle. \end{aligned} \tag{2.8}$$

† The study of the cohomology of the general graded contractions of $so(N+1)$ could be performed similarly; we shall nevertheless restrict ourselves to the CK family. This means, for instance, that the 'completely contracted' (and hence Abelian) algebra, for which $\dim H^2(\mathcal{G}, \mathbb{R}) = \binom{\dim \mathcal{G}}{2}$, is not included.

If either $a = 1$ or $a = N$, then the direct sum in (2.7) has a single summand. This decomposition as a semidirect sum is true irrespective of whether the remaining constants ω_i are equal to zero or not.

The structure behind this decomposition can be described visually by setting the generators in a triangular array. The generators spanning the subspace t are those inside the rectangle, while the subalgebras $so_{\omega_1,\ldots,\omega_{a-1}}(a)$ and $so_{\omega_{a+1},\ldots,\omega_N}(N+1-a)$ correspond to the two triangles to the left and below the rectangle respectively,

$$
\begin{array}{cccc|cccc}
\Omega_{01} & \Omega_{02} & \ldots & \Omega_{0\,a-1} & \Omega_{0a} & \Omega_{0\,a+1} & \ldots & \Omega_{0N} \\
 & \Omega_{12} & \ldots & \Omega_{1\,a-1} & \Omega_{1a} & \Omega_{1\,a+1} & \ldots & \Omega_{1N} \\
 & & \ddots & \vdots & \vdots & \vdots & & \vdots \\
 & & & \Omega_{a-2\,a-1} & \Omega_{a-2\,a} & \Omega_{a-2\,a+1} & \ldots & \Omega_{a-2\,N} \\
 & & & & \Omega_{a-1\,a} & \Omega_{a-1\,a+1} & \ldots & \Omega_{a-1\,N} \\
\cline{5-8}
\multicolumn{1}{c}{} & & & \multicolumn{1}{c}{} & & \Omega_{a\,a+1} & \ldots & \Omega_{aN} \\
\multicolumn{1}{c}{} & & & \multicolumn{1}{c}{} & & & \ddots & \vdots \\
\multicolumn{1}{c}{} & & & \multicolumn{1}{c}{} & & & & \Omega_{N-1\,N}.
\end{array}
$$

The Abelian rectangle is reduced to the row Ω_{0i}, $i = 1, \ldots, N$ for $\omega_1 = 0$ or to the column Ω_{iN}, $i = 0, 1, \ldots, N-1$ when $\omega_N = 0$.

Let us consider some specially relevant algebras in the CK family. All ω's are assumed to be different from zero, unless otherwise explicitly stated.

(1) $\omega_a \neq 0\ \forall a$. Here $so_{\omega_1,\ldots,\omega_N}(N+1)$ is a pseudo-orthogonal algebra $so(p,q)$ with $p+q = N+1$. The matrix representation (2.5) of this algebra generates a group of linear transformations on $N+1$ real variables which leave invariant a quadratic form g, with matrix $\mathrm{diag}(1, \omega_{01}, \omega_{02}, \ldots, \omega_{0N})$. The signature of this quadratic form is the number of positive and negative terms in the sequence $(1, \omega_1, \omega_1\omega_2, \ldots, \omega_1\omega_2\ldots\omega_N)$ so that each ω_i governs the relative sign of two consecutive diagonal elements in the metric matrix, the element where ω_i appears for the first time and the previous one.

(2) $\omega_1 = 0$. The $so_{0,\omega_2,\ldots,\omega_N}(N+1)$ algebras are the usual pseudo-orthogonal inhomogeneous ones, with a semidirect sum structure given by

$$so_{0,\omega_2,\ldots,\omega_N}(N+1) \equiv t_N \odot so_{\omega_2,\ldots,\omega_N}(N) \equiv iso(p,q) \qquad p+q = N$$

where $so_{\omega_2,\ldots,\omega_N}(N)$ acts on t_N through the vector representation. The Euclidean algebra $iso(N)$ appears once with $(\omega_1, \omega_2, \ldots, \omega_N) = (0, 1, \ldots, 1)$. The Poincaré algebra $iso(N-1, 1)$ is reproduced several times, e.g. for $(\omega_1, \omega_2, \ldots, \omega_N) = (0, -1, 1, \ldots, 1)$.

(3) $\omega_1 = \omega_2 = 0$. These algebras have two different semidirect sum structures (cf (2.7)). The one associated with the vanishing of ω_1 is

$$t_N \odot so_{\omega_2=0,\omega_3,\ldots,\omega_N}(N)$$

while the structure associated with the vanishing of ω_2 is

$$t_{2N-2} \odot (so_{\omega_1=0}(2) \oplus so_{\omega_3,\ldots,\omega_N}(N-1)).$$

The first structure can also be seen as a 'twice-inhomogeneous' pseudo-orthogonal algebra

$$so_{0,0,\omega_3,\ldots,\omega_N}(N+1) \equiv t_N \odot (t_{N-1} \odot so_{\omega_3,\ldots,\omega_N}(N-1)) \equiv iiso(p,q) \qquad p+q = N-1. \tag{2.9}$$

For example, the Galilean algebra $iiso(N-1)$ appears for $\omega_i = (0, 0, 1, \ldots, 1)$. This pattern continues for $\omega_1 = \omega_2 = \omega_3 = 0$, etc.

(4) $\omega_N = 0$. Here the algebras have a semidirect sum structure:

$$so_{\omega_1,\omega_2,\ldots,\omega_{N-1},0}(N+1) \equiv t'_N \odot so_{\omega_1,\omega_2,\ldots,\omega_{N-1}}(N) \equiv i'so(p,q) \tag{2.10}$$

where now $so(p,q)$ acts on t'_N through the contragredient of the vector representation, hence the notation with a prime. Of course these algebras are isomorphic to the ones described in (2) as above (cf (2.6)). A pattern similar to that in (3) occurs for the cases $\omega_N = \omega_{N-1} = 0$, etc.

(5) $\omega_1 = \omega_N = 0$. They have two different semidirect sum splittings. The first is

$$so_{0,\omega_2,\ldots,\omega_{N-1},0}(N+1) \equiv t_N \odot (t'_{N-1} \odot so_{\omega_2,\ldots,\omega_{N-1}}(N-1)) \equiv ii'so(p,q) \tag{2.11}$$

where $p+q = N-1$; $so(p,q)$ acts on t'_{N-1} through the contragredient of the vector representation while $i'so(p,q)$ acts on t_N through the vector representation. The other is:

$$so_{0,\omega_2,\ldots,\omega_{N-1},0}(N+1) \equiv t'_N \odot (t_{N-1} \odot so_{\omega_2,\ldots,\omega_{N-1}}(N-1)) \equiv i'iso(p,q). \tag{2.12}$$

One example of (2.11) is $ii'so(3)$, the Carroll algebra in $(3+1)$ dimensions [16], which corresponds to $(0,1,1,0)$.

(6) $\omega_a = 0$, $a \neq 1, N$. The structure of these algebras can be schematically described as $t_r \odot (so(p,q) \oplus so(p',q'))$ (see [17]). In particular, for $\omega_2 = 0$ we have $t_{2N-2} \odot (so(p,q) \oplus so(p',q'))$ with $p+q = 2$ and $p'+q' = N-1$, which include for $q' = 0$ the oscillating and expanding Newton–Hooke algebras [16] associated with $(1,0,1,\ldots,1)$ and $(-1,0,1,\ldots,1)$, respectively.

(7) The fully contracted case in the CK family corresponds to setting all constants $\omega_a = 0$. This is the so-called flag algebra $so_{0,\ldots,0}(N+1) \equiv i\ldots iso(1)$ [11].

The kinematical algebras associated with different models of spacetime [16] belong to the family of CK algebras, and this indeed provides one of the strongest physical motivations to study this family of algebras; in relation with the graded-contraction point of view, see also [18] and [10].

3. Central extensions of the CK algebras

Our aim in this section is to obtain the general solution to the problem of finding all central extensions for all the CK algebras and in arbitrary dimensions.

We write the independent commutation relations of $so_{\omega_1,\ldots,\omega_N}(N+1)$ (2.4) as

$$[\Omega_{ab}, \Omega_{cd}] = \sum_{\substack{i,j=0\\ i<j}}^{N} C^{ij}_{ab,cd}\Omega_{ij} \tag{3.1}$$

where, as before, in any Ω_{ef}, $e < f$ is always assumed. The four types of structure constants in (2.4) are given by

$$C^{ij}_{ab,ac} = \delta^i_b\delta^j_c\omega_{ab} \qquad C^{ij}_{ab,bc} = -\delta^i_a\delta^j_c \qquad C^{ij}_{ac,bc} = \delta^i_a\delta^j_b\omega_{bc} \qquad a<b<c \tag{3.2}$$

together with $C^{ij}_{ab,cd} = 0$ if all a, b, c, d are different.

Any central extension $\overline{so}_{\omega_1,\ldots,\omega_N}(N+1)$ of the algebra $so_{\omega_1,\ldots,\omega_N}(N+1)$ by the one-dimensional algebra of generator Ξ will have generators (Ω_{ab}, Ξ) and commutators:

$$[\Omega_{ab}, \Omega_{cd}] = \sum_{\substack{i,j=0\\ i<j}}^{N} C^{ij}_{ab,cd}\Omega_{ij} + \alpha_{ab,cd}\,\Xi \qquad [\Xi, \Omega_{ab}] = 0 \tag{3.3}$$

where the extension coefficients to be determined $\alpha_{ab,cd}$ (‘central charges’), must be antisymmetric in the interchange of pairs ab and cd,

$$\alpha_{cd,ab} = -\alpha_{ab,cd} \tag{3.4}$$

and must fulfil the conditions

$$\sum_{\substack{i,j=0\\ i<j}}^{N} (C^{ij}_{ab,cd}\alpha_{ij,ef} + C^{ij}_{cd,ef}\alpha_{ij,ab} + C^{ij}_{ef,ab}\alpha_{ij,cd}) = 0 \qquad (3.5)$$

which follow from the Jacobi identity. The 'extension coefficients' are the coordinates $(\alpha(\Omega_{ab}, \Omega_{cd}) = \alpha_{ab,cd})$ of the antisymmetric rank-two tensor α which is the two-cocycle of the specific extension being considered and (3.5) is the two-cocycle condition for the Lie-algebra cohomology. The classes of non-trivial two-cocycles associated with the tensors α determine the dimension of the second cohomology group $H^2(\mathcal{G}, \mathbb{R})$. The rest of this section will be devoted to characterizing first the vector space of all tensors α satisfying conditions (3.4) and (3.5) for the CK algebra $so_{\omega_1,\ldots,\omega_N}(N+1)$. Second, the question of the possible equivalence of two extensions given by two tensors α will be addressed and solved. The reader who is not interested in the details of the calculation procedure may skip the rest of this section; its results are summarized in theorem 4.1.

3.1. Setting up the problem: Jacobi identities

The antisymmetry of $\alpha_{ab,cd}$ will be automatically taken into account by considering as independent coefficients only those $\alpha_{ab,cd}$ with $a \leqslant c$ and $b < d$ when $a = c$ (the conditions $a < b$ and $c < d$ are always assumed).

The first step consists of solving the $\binom{(N+1)N/2}{3}$ (see below) Jacobi identities (3.5) understood as equations in the coefficients $\alpha_{ab,cd}$. There is one equation for each possible set of three index pairs ab, cd, ef in (3.5) and looking at how many of these indices are *different* we can group all Jacobi identities into four classes.

• One equation for the only possible set (ab, ac, bc) of six indices made up from three different indices (permutations will lead to the same Jacobi identity).

• 16 equations, one for each possible set of six indices made up from four different indices

$$\begin{aligned}
&(ab, ac, ad), (ab, ac, bd), (ab, bc, ad), (ab, ac, cd), (ac, ad, bc), (ab, ad, cd),\\
&(ac, ad, bd), (ab, bc, bd), (ab, bc, cd), (ac, bc, bd), (ab, bd, cd), (ad, bd, bc),\\
&(ac, bc, cd), (ac, bd, cd), (ad, bc, cd), (ad, bd, cd).
\end{aligned} \qquad (3.6)$$

• 30 equations, one for each possible set of six indices made up from five different indices

$$\begin{aligned}
&(ab, ac, de), (ab, ad, ce), (ab, ae, cd), (ac, ae, bd), (ac, ad, be), (ad, ae, bc),\\
&(ab, bc, de), (ab, bd, ce), (ab, be, cd), (ad, bc, be), (ac, bd, be), (ae, bc, bd),\\
&(ab, cd, ce), (ac, bc, de), (ad, bc, ce), (ae, bc, cd), (ac, bd, ce), (ac, be, cd),\\
&(ab, cd, de), (ac, bd, de), (ae, bd, cd), (ad, bc, de), (ad, bd, ce), (ad, be, cd),\\
&(ab, ce, de), (ac, be, de), (ae, bc, de), (ad, be, ce), (ae, be, cd), (ae, ce, bd).
\end{aligned} \qquad (3.7)$$

• 15 equations, one for each possible set of six indices $(ab, cd, ef), (ab, ce, df), \ldots,$ made up from six different indices.

The following relation, where it is understood that $\binom{m}{n} = 0$ if $m < n$, checks the above splitting:

$$\binom{(N+1)N/2}{3} = 1\binom{N+1}{3} + 16\binom{N+1}{4} + 30\binom{N+1}{5} + 15\binom{N+1}{6}. \qquad (3.8)$$

Now we write explicitly the above equations. To begin with, the equation involving only three different indices as well as the 15 equations with six different indices are easily seen to be trivially satisfied due to (3.4) or to the fact that $C^{ij}_{ab,cd} = 0$ whenever (ab, cd) are all different indices. The 16 Jacobi identities for four indices $a < b < c < d$ lead to 10 equations, written here in bulk (later we shall write these equations in a neater way):

$$\begin{array}{ll} \alpha_{ab,ad} = -\omega_{ab}\alpha_{bc,cd} & \alpha_{ac,ad} = \omega_{ab}\alpha_{bc,bd} \\ \alpha_{ad,cd} = -\omega_{cd}\alpha_{ab,bc} & \alpha_{ad,bd} = \omega_{cd}\alpha_{ac,bc} \\ \alpha_{ac,cd} = \alpha_{ab,bd} & \omega_{cd}\alpha_{ab,ac} = \omega_{ab}\alpha_{bd,cd} \\ \alpha_{ac,bd} = -\omega_{bc}\alpha_{ab,cd} & \alpha_{ad,bc} = 0 \\ \omega_{ac}\alpha_{ab,cd} = 0 & \omega_{bd}\alpha_{ab,cd} = 0. \end{array} \tag{3.9}$$

(The 16 equations involve several pairs $\omega\alpha = 0$ and $\omega\omega'\alpha = 0$; in these cases the second is clearly a consequence of the first, and may therefore be discarded.) On their part, the 30 Jacobi identities for five indices $a < b < c < d < e$, give rise to sixteen equations:

$$\begin{array}{lll} \alpha_{ab,ce} = 0 & & \\ \alpha_{ac,be} = \alpha_{ac,de} = 0 & & \\ \alpha_{ad,be} = \alpha_{ad,ce} = 0 & & \\ \alpha_{ae,bd} = \alpha_{ae,bc} = \alpha_{ae,cd} = 0 & & \\ \omega_{bc}\alpha_{ab,de} = 0 & \omega_{cd}\alpha_{ab,de} = 0 & \omega_{de}\alpha_{ab,cd} = 0 \\ \omega_{de}\alpha_{ac,bd} = 0 & \omega_{de}\alpha_{ad,bc} = 0 & \\ \omega_{ab}\alpha_{bc,de} = 0 & \omega_{ab}\alpha_{bd,ce} = 0 & \omega_{ab}\alpha_{be,cd} = 0. \end{array} \tag{3.10}$$

3.2. Solving strategy

The structure behind equations (3.9) and (3.10) is not readily apparent. In order to unveil this structure and solve the central extension problem, we shall;

(1) sort out all extension coefficients (coordinates of α) into disjoint classes,

(2) group all equations as related to the former classification and isolate coefficients which can be simply expressed in terms of the remaining ones,

(3) state the form of the general solution in terms of basic extension coefficients from which all others are derived, but which still may be subjected to some additional relations, and

(4) analyse when an extension is trivial (or when two extensions can be equivalent).

It is convenient to put the coordinates of the generic α into three *types*.

Type I. Coefficients $\alpha_{ab,bc}$ with *three* different indices $a < b < c$ with the *middle* index common to both pairs.

Types IIF/IIL. Coefficients $\alpha_{ab,ac}/\alpha_{ac,bc}$, with *three* different indices $a < b < c$ where the *first/last* index is common to both pairs.

Type III. Coefficients $\alpha_{ab,cd}$ with *four* different indices $a < b$, $a < c < d$.

Before studying them below, it may be worthwhile advancing now that, for all CK algebras, those of type I will always correspond to two-coboundaries. Those of type IIF/IIL will determine two-cocycles (which may be trivial) which result from the contraction of two-coboundaries through the pseudoextension mechanism [19, 20] and those of type III will determine non-trivial two-cocycles which are not generated by contraction from two-coboundaries (i.e. different from those in IIF/IIL).

The Jacobi equations (3.5) include a block of relations such as (3.9) for each set of *four* different indices $a < b < c < d$ and a group of equations (3.10) for each set of *five* indices $a < b < c < d < e$. In order to deal with these equations simultaneously, it will be convenient to start with any set of five indices and to consider jointly equations (3.10) and five copies of equations (3.9), one for every choice of four indices out of the five $abcde$, namely $abcd$, $abce$, $abde$, $acde$ and $bcde$. The complete set of equations thus obtained involves the 45 central extension coefficients $\alpha_{ab,cd}, \alpha_{ab,ce}, \ldots$ with indices in the set $\{a, b, c, d, e\}$ (there are $\binom{5}{3} = 10$ type I coefficients, $10 + 10$ type IIF + IIL coefficients and 15 (see (3.10)) of type III). We now write all equations (assuming $a < b < c < d < e$) and group them in a convenient way. Out of these 45 coefficients, 30 are either equal to zero or can be expressed by simple relations in terms of the remaining extension coefficients. These 30 coefficients are called *abcde-derived*, and are related to the remaining 15 *abcde-primary* coefficients by the equations below, the left/right-hand sides of which involve only derived/primary coefficients:

I/I $\quad \alpha_{ac,cd} = \alpha_{ab,bd} \qquad \alpha_{ac,ce} = \alpha_{ab,be} \qquad \alpha_{ad,de} = \alpha_{ab,be} \qquad \alpha_{bd,de} = \alpha_{bc,ce}$ (3.11)

IIF/IIF $\quad \alpha_{ac,ad} = \omega_{ab}\alpha_{bc,bd} \qquad \alpha_{ad,ae} = \omega_{ac}\alpha_{cd,ce} \qquad \alpha_{bd,be} = \omega_{bc}\alpha_{cd,ce}$ (3.12)

IIL/IIL $\quad \alpha_{ae,be} = \omega_{ce}\alpha_{ac,bc} \qquad \alpha_{ad,bd} = \omega_{cd}\alpha_{ac,bc} \qquad \alpha_{be,ce} = \omega_{de}\alpha_{bd,cd}$ (3.13)

IIF/I $\quad \alpha_{ab,ad} = -\omega_{ab}\alpha_{bc,cd} \qquad \alpha_{ab,ae} = -\omega_{ab}\alpha_{bc,ce}$

$\qquad \alpha_{ac,ae} = -\omega_{ac}\alpha_{cd,de} \qquad \alpha_{bc,be} = -\omega_{bc}\alpha_{cd,de}$ (3.14)

IIL/I $\quad \alpha_{ad,cd} = -\omega_{cd}\alpha_{ab,bc} \qquad \alpha_{ae,ce} = -\omega_{ce}\alpha_{ab,bc}$

$\qquad \alpha_{ae,de} = -\omega_{de}\alpha_{ab,bd} \qquad \alpha_{be,de} = -\omega_{de}\alpha_{bc,cd}$ (3.15)

III/III $\quad \alpha_{ac,bd} = -\omega_{bc}\alpha_{ab,cd} \qquad \alpha_{bd,ce} = -\omega_{cd}\alpha_{bc,de}$ (3.16)

$\qquad \alpha_{ab,ce} = \alpha_{ac,be} = \alpha_{ae,bc} = 0 \qquad \alpha_{ad,bc} = 0$

III $\quad \alpha_{ae,bd} = \alpha_{ad,be} = 0 \qquad \alpha_{be,cd} = 0$ (3.17)

$\qquad \alpha_{ac,de} = \alpha_{ad,ce} = \alpha_{ae,cd} = 0.$

Sorted out by their type, the 15 $abcde$-primary coefficients are:

I	$\alpha_{ab,bc}$	$\alpha_{ab,bd}$	$\alpha_{ab,be}$	$\alpha_{bc,cd}$	$\alpha_{bc,ce}$	$\alpha_{cd,de}$
IIF	$\alpha_{ab,ac}$	$\alpha_{bc,bd}$	$\alpha_{cd,ce}$			
IIL	$\alpha_{ac,bc}$	$\alpha_{bd,cd}$	$\alpha_{ce,de}$			
III	$\alpha_{ab,cd}$	$\alpha_{ab,de}$	$\alpha_{bc,de}$.			

(3.18)

These primary coefficients are themselves constrained by the relations:

IIF/IIL $\quad \omega_{cd}\alpha_{ab,ac} = \omega_{ab}\alpha_{bd,cd} \qquad \omega_{de}\alpha_{bc,bd} = \omega_{bc}\alpha_{ce,de}$ (3.19)

$\qquad \omega_{ac}\alpha_{ab,cd} = 0 \qquad \omega_{bd}\alpha_{ab,cd} = 0 \qquad \omega_{de}\alpha_{ab,cd} = 0$

III $\quad \omega_{bc}\alpha_{ab,de} = 0 \qquad \omega_{cd}\alpha_{ab,de} = 0$ (3.20)

$\qquad \omega_{ab}\alpha_{bc,de} = 0 \qquad \omega_{bd}\alpha_{bc,de} = 0 \qquad \omega_{ce}\alpha_{bc,de} = 0$

which follow from (3.9) and (3.10). Note that since the ω's may be zero, we cannot express any of these primary coefficients in terms of the others.

We have indicated the type of each group of coefficients, by the corresponding symbol in each line, e.g. the rows in (3.18) correspond to the three types (I, IIF/IIL, III) as given above. Summing up, the 15 *abcde-primary* extension coefficients are as follows.

• Those type I with the first pair $abcde$-contiguous (i.e. the indices in the first pair are consecutive in the ordered sequence $abcde$).

• Those type IIF/L with the first/last pair of indices $abcde$-contiguous and three $abcde$-consecutive indices (of course, the later condition implies the former).

- Type III with two $abcde$-contiguous pairs.

All others are $abcde$-derived.

3.3. The basic extension coefficients

The next step in this process is to consider the above results for all possible numerical values of the five indices $abcde$ since these numerical values appear as the indices labelling the coordinates of the tensor α. Consider, for instance, the coordinate $\alpha_{13,14}$ in a case with, say, $N = 8$. This is a 13457-primary one, which means that it cannot be expressed in terms of another coefficient with indices taken from the set 13457. However, the same coefficient appears as a derived one for the set of indices 12345, because the first index pair 13 are not contiguous indices in the set 12345. It is clear that only those coefficients with the form of (3.18) for *all* choices of five indices will be the really primary, or basic, ones; all other can be ultimately derived in terms of these. We shall call them *basic coefficients* of the extension and introduce a notation which highlights their role in the extensions. By checking the former list of $abcde$-primary extension coefficients, we readily conclude that

Proposition 3.1. The basic coefficients of the extension are as follows.

- Type I with the first pair of indices contiguous

$$\tau_{ac} := \alpha_{a\,a+1,a+1\,c} \qquad a = 0, 1, \ldots, N-2 \qquad c = a+2, \ldots, N \qquad N \geqslant 2. \tag{3.21}$$

We remark that the indices in τ_{ac} cannot be consecutive; there are $N(N-1)/2$ basic type I τ_{ac} extension coefficients.

- Type IIF/IIL with three consecutive indices (and therefore with the first/last pair of contiguous indices):

$$\alpha^F_{a+1\,a+2} := \alpha_{a\,a+1,a\,a+2} \qquad a = 0, 1, \ldots, N-2 \qquad N \geqslant 2. \tag{3.22}$$

$$\alpha^L_{a-2\,a-1} := \alpha_{a-2\,a,a-1\,a} \qquad a = 2, \ldots, N \qquad N \geqslant 2. \tag{3.23}$$

There are $(N-1)$ basic type II extension coefficients for each subtype IIF and IIL.

- Type III with two contiguous pairs of indices:

$$\beta_{b+1\,d+1} := \alpha_{b\,b+1,d\,d+1} \qquad b = 0, 1, \ldots, N-3 \qquad d = b+2, \ldots, N-1 \qquad N \geqslant 3. \tag{3.24}$$

These β extension coefficients must have two not consecutive indices, and the index 0 cannot appear in any β. The possible number of these extension coefficients is $(N-1)(N-2)/2$.

These basic coefficients are still not independent and must fulfil the Jacobi relations (3.19) and (3.20). In particular, for basic type II coefficients, (3.19) now reads:

$$\text{IIF/IIL} \qquad \omega_{a+3}\alpha^F_{a+1\,a+2} = \omega_{a+1}\alpha^L_{a+1\,a+2} \qquad a = 0, \ldots, N-3 \tag{3.25}$$

and (3.20) for basic type III coefficients reduces to either

$$\omega\beta_{b+1\,b+3} = 0 \qquad \text{for } \omega = \omega_b, \omega_{b+1}\omega_{b+2}, \omega_{b+2}\omega_{b+3}, \omega_{b+4} \tag{3.26}$$

where for $b = 0/b = N-3$ the first/last condition $\omega_b\beta = 0/\omega_{b+4}\beta = 0$ (which would read $\omega_0\beta = 0/\omega_{N+1}\beta = 0$) is not present, or to

$$\omega\beta_{b+1\,d+1} = 0 \qquad \text{for } \omega = \omega_b, \omega_{b+2}, \omega_d, \omega_{d+2} \tag{3.27}$$

where $b+1$ and $d+1$ are not next neighbours, with similar restrictions as before for the extreme equations.

The basic type II coefficients are $\alpha_{a\,a+1,a\,a+2}$ and $\alpha_{a+1\,a+3,a+2\,a+3}$. Both appear in the extended commutators:

$$\begin{aligned}
[\Omega_{a\,a+1}, \Omega_{a\,a+2}] &= \omega_{a+1}\Omega_{a+1\,a+2} + \alpha_{a\,a+1,a\,a+2}\,\Xi \\
[\Omega_{a+1\,a+3}, \Omega_{a+2\,a+3}] &= \omega_{a+3}\Omega_{a+1\,a+2} + \alpha_{a+1\,a+3,a+2\,a+3}\,\Xi
\end{aligned} \tag{3.28}$$

so both extension coefficients appear related to the generator $\Omega_{a+1\,a+2}$, which explains the notations $\alpha^F_{a+1\,a+2} := \alpha_{a\,a+1,a\,a+2}$ and $\alpha^L_{a+1\,a+2} := \alpha_{a+1\,a+3,a+2\,a+3}$ in (3.22) and (3.23). Type II basic coefficients are grouped in a *single* coefficient, α^L_{01}, $(N-2)$ *pairs*, $\alpha^F_{12}, \alpha^L_{12}, \ldots, \alpha^F_{N-2\,N-1}, \alpha^L_{N-2\,N-1}$ and another *single* coefficient, $\alpha^F_{N-1\,N}$, the single ones appearing for the cases where the index pair does not have a predecessor or a successor. Type III basic extension coefficients $\alpha_{b\,b+1,d\,d+1}$ appear in the extended commutators,

$$[\Omega_{b\,b+1}, \Omega_{d\,d+1}] = \alpha_{b\,b+1,d\,d+1}\,\Xi. \tag{3.29}$$

Let us comment on the process of finding the derived coefficients in terms of the basic extension coefficients. For type I coefficients, the only constraint is equation (3.11), which says that all type I coefficients $\alpha_{ab,bc}$ with the same ac indices are simply equal; this is $\alpha_{ab,bc} = \alpha_{aa+1,a+1c} = \tau_{ac}$. Consider now the derived type IIF coefficients. They must have three non-consecutive indices, and there are three possibilities, representatives of which are the coefficients $\alpha_{ab,ad}$ (when ab are contiguous but bd are not), $\alpha_{ac,ad}$ (when ac are not contiguous but cd are) and $\alpha_{ac,ae}$ (when neither ac nor ce are contiguous). For the first one, an equation in (3.14) gives $\alpha_{ab,ad} = -\omega_{ab}\alpha_{bc,cd}$ and choosing $c = b+1$ we obtain $\alpha_{ab,ad} = -\omega_{ab}\alpha_{b\,b+1,b+1\,d} = -\omega_{ab}\tau_{bd}$; as in this case $b = a+1$ we obtain $\alpha_{a\,a+1,ad} = -\omega_{a+1}\tau_{a+1\,d}$ for $d = a+3, \ldots, N$. For the second, one of equations (3.12) with $b = c-1$ gives $\alpha_{ac,ad} = \omega_{a\,c-1}\alpha_{c-1\,c,c-1\,d}$; now as here $d = c+1$, and the extension coefficient $\alpha_{c-1\,c,c-1\,d}$ is equal to $\alpha^F_{c\,c+1}$, we finally obtain $\alpha_{ac,a\,c+1} = \omega_{a\,c-1}\alpha^F_{c\,c+1}$ for $c = a+2, \ldots, N-1$. In the third case, we use one of the equations in (3.14) with $d = c+1$ to obtain directly $\alpha_{ac,ae} = -\omega_{ac}\alpha_{c\,c+1,c+1\,e} = -\omega_{ac}\tau_{ce}$ for $c = a+2, \ldots, N-2$ and $e = c+2, \ldots, N$. A completely similar process gives the derived type IIL.

Finally, type III has a single class of derived coefficients which might be different from zero: $\alpha_{ac,bd}$ where ab, bc and cd are contiguous pairs (so all four $abcd$ indices are consecutive, say $a\,a{+}1\,a{+}2\,a{+}3$). These are given by equations (3.16) in term of the basic ones as $\alpha_{a\,a+2,a+1\,a+3} = -\omega_{a+1\,a+2}\alpha_{a\,a+1,a+2\,a+3} = -\omega_{a+2}\beta_{a+1\,a+3}$. All other non-basic type III coefficients are necessarily equal to zero.

To sum up, for $N = 2$ there are no derived extension coefficients; for any $N \geqslant 3$, the complete list of derived extension coefficients is therefore given by the following proposition.

Proposition 3.2. For $N \geqslant 3$, the derived extension coefficients are as follows.

- Type I, with the first pair non-contiguous

$$\alpha_{ac,cd} = \tau_{ad} \qquad a = 0, 1, \ldots, N-3 \qquad c = a+2, \ldots, N-1 \qquad d = c+1, \ldots, N. \tag{3.30}$$

- Type IIF/IIL, with three non-consecutive indices. There are three possibilities, according to whether the first and second indices or the second and third are or not contiguous:

$$\begin{array}{lll}
\alpha_{a\,a+1,ad} = -\omega_{a+1}\tau_{a+1\,d} & a = 0, 1, \ldots, N-3 & d = a+3, \ldots, N \\
\alpha_{ac,a\,c+1} = \omega_{a\,c-1}\alpha^F_{c\,c+1} & a = 0, 1, \ldots, N-3 & c = a+2, \ldots, N-1 \\
\alpha_{ac,ae} = -\omega_{ac}\tau_{ce} & a = 0, 1, \ldots, N-4 & c = a+2, \ldots, N-2 \\
 & e = c+2, \ldots, N & (\text{here } N \geqslant 4)
\end{array} \tag{3.31}$$

$$\begin{array}{lll}
\alpha_{a\,c+1,c\,c+1} = -\omega_{c+1}\tau_{ac} & a = 0, 1, \ldots, N-3 & c = a+2, \ldots, N-1 \\
\alpha_{ac,a+1\,c} = \omega_{a+2\,c}\alpha^{L}_{a\,a+1} & a = 0, 1, \ldots, N-3 & c = a+3, \ldots, N \\
\alpha_{ae,ce} = -\omega_{ce}\tau_{ac} & a = 0, 1, \ldots, N-4 & c = a+2, \ldots, N-2 \\
 & e = c+2, \ldots, N & (\text{here } N \geqslant 4).
\end{array} \tag{3.32}$$

Type III, with at least a non-contiguous pair. Only those of the form $\alpha_{ac,bd}$ with $abcd$ consecutive are possibly different from zero, and are given by

$$\alpha_{a\,a+2,a+1\,a+3} = -\omega_{a+2}\,\beta_{a+1\,a+3} \qquad a = 0, 1, \ldots, N-3 \tag{3.33}$$

all other non-basic type III extension coefficients are necessarily equal to zero.

It can be checked that for any choice of the extension coefficients (satisfying the equations (3.25), (3.26) and (3.27)) the expressions given above for the derived extension coefficients satisfy all Jacobi equations. This is cumbersome but straightforward and will not be done here.

3.4. Equivalence of extensions: two-coboundaries

So far we have determined the general form of a two-cocycle on the CK algebra $so_{\omega_1,\ldots,\omega_N}(N+1)$. Two two-cocycles differing by a two-coboundary lead to equivalent extensions, so the next step is to find the general form of a coboundary. Let us make the change of generators $\Omega_{ab} \to \Omega'_{ab} = \Omega_{ab} + \mu_{ab}\,\Xi$, where μ_{ab} are arbitrary real numbers. The commutation relations for the new generators Ω'_{ab}, obtained from (3.3) with a given two-cocycle $\alpha_{ab,cd}$ are:

$$[\Omega'_{ab}, \Omega'_{cd}] = \sum_{i,j=0}^{N} C^{ij}_{ab,cd}\Omega'_{ij} + \left(\alpha_{ab,cd} - \sum_{i,j=0}^{N} C^{ij}_{ab,cd}\mu_{ij}\right)\Xi. \tag{3.34}$$

Therefore, the general expression of a two-coboundary $\delta\mu$ generated by μ is

$$(\delta\mu)_{ab,cd} = \sum_{i,j=0}^{N} C^{ij}_{ab,cd}\mu_{ij}. \tag{3.35}$$

Using the expressions (3.2) for the structure constants, we obtain

$$\begin{array}{ll}
\text{I} & (\delta\mu)_{ab,bc} = -\mu_{ac} \\
\text{IIF/IIL} & (\delta\mu)_{ab,ac} = \omega_{ab}\mu_{bc} \;/\; (\delta\mu)_{ac,bc} = \omega_{bc}\mu_{ab} \\
\text{III} & (\delta\mu)_{ab,cd} = (\delta\mu)_{ac,bd} = (\delta\mu)_{ad,bc} = 0.
\end{array} \tag{3.36}$$

The question of whether the previously found extension coefficients (or two-cocycles) define trivial central extensions amounts to checking whether they have the form of a two-coboundary, (3.36), which may then be used to eliminate the central Ξ term from (3.34). This depends on the vanishing of the constants ω_i. In fact, the previous analysis classifies the extensions into three types, which behave in three different ways.

• Type I extensions can be carried out for all CK algebras, as there are no any ω_i-dependent restrictions for the basic type I coefficients τ_{ac}. However these extensions are always trivial (for all CK algebras simultaneously, as seen in (3.36)), and will be discarded. All expressions simplify considerably if we take this into account, as we shall do from now on. This 'uses up' those coboundaries coming from the values μ_{ac} with two non-consecutive ac indices. Further equivalences (already for type II) are restricted to redefinitions of generators with two consecutive indices, $\Omega_{a\,a+1} \to \Omega'_{a\,a+1} = \Omega_{a\,a+1} + \mu_{a\,a+1}\Xi$ (see (3.37) and (4.4) below).

• Type II coefficients can appear in all CK algebras, as the ω_i-dependent restrictions (3.25) are not strong enough to force all these coefficients to vanish. However, the triviality of these extensions is also ω_i-dependent, and we will see that the $2(N-1)$ extensions corresponding to the basic extension coefficients $\alpha^L_{01}, \alpha^F_{12}, \alpha^L_{12}, \ldots, \alpha^F_{N-2\,N-1}, \alpha^L_{N-2\,N-1}, \alpha^F_{N-1\,N}$ are all trivial for the simple algebras and all non-trivial for the extreme case of the flag algebra. It is within this particular type of extensions that a *pseudo-extension* (trivial extension by a two-coboundary) may become a non-trivial extension by contraction.

• Type III coefficients behave in a completely different way. The ω_i-dependent restrictions (3.26) and (3.27) on type III basic coefficients force many of these coefficients to vanish (depending on how many constants ω_i are equal to zero). Those remaining, once they are present (that is, allowed) are always non-trivial. This means that there are no type III non-trivial central extensions coming by contraction from pseudo-extensions.

All that remains is to discuss the possible equivalence among type II extensions. The basic type II values of the coboundary associated with the change of generators $\Omega_{a+1\,a+2} \to \Omega'_{a+1\,a+2} = \Omega_{a+1\,a+2} + \mu_{a+1\,a+2}\Xi$ are

$$\text{IIF/IIL} \qquad (\delta\mu)^F_{a+1\,a+2} = \omega_{a+1}\mu_{a+1\,a+2} \qquad (\delta\mu)^L_{a+1\,a+2} = \omega_{a+3}\mu_{a+1\,a+2}. \tag{3.37}$$

We remark that (as it should) these coboundaries automatically satisfy equation (3.25). We must study now how the freedom afforded by these changes can be used to reduce to zero some of the extension coefficients.

Consider first the single α^L_{01}, the value of which can be arbitrary. We see in (3.37) that as long as $\omega_2 \neq 0$, we can reduce it to zero by using the coboundary μ_{01}. Then the extension corresponding to the basic coefficient α^L_{01} is non-trivial when $\omega_2 = 0$ and trivial otherwise. Likewise, the extension corresponding to the basic coefficient $\alpha^F_{N-1\,N}$ is non-trivial when $\omega_{N-1} = 0$ and trivial otherwise. The possible triviality of these extensions is thus completely governed by two constants ω_i which play a special role: the second ω_2 and the last but one ω_{N-1}.

Let us now look at the case of pairs $\alpha^F_{a+1\,a+2}, \alpha^L_{a+1\,a+2}$. Here the situation is controlled by two constants ω_i, namely ω_{a+1} and ω_{a+3}. When they are both equal to zero, equation (3.25) is automatically satisfied, irrespective of the values of the pair of coefficients $\alpha^F_{a+1\,a+2}$ and $\alpha^L_{a+1\,a+2}$ which cannot be modified by adding a coboundary; in this case the cocycles associated with these coefficients are simultaneously non-trivial. When only one of the constants ω_{a+1} and ω_{a+3} is equal to zero, equation (3.25) forces the vanishing of one of the coefficients in the pair, while the other can be reduced to zero by the appropriate coboundary coming from $\mu_{a+1\,a+2}$. Finally, when both constants ω_{a+1} and ω_{a+3} are different from zero, then (3.25) enforces the possibility of simultaneously reducing $\alpha^F_{a+1\,a+2}$ and $\alpha^L_{a+1\,a+2}$ to zero by using the coboundary coming from $\mu_{a+1\,a+2}$. Therefore the two two-cocycles extensions corresponding to the two basic coefficients $\alpha^F_{a+1\,a+2}$ and $\alpha^L_{a+1\,a+2}$ are non-trivial when both $\omega_{a+1} = 0$ and $\omega_{a+3} = 0$; the two extensions are simultaneously trivial otherwise.

Once the coboundary type I coefficients are removed, the contents of propositions 3.1 and 3.2 may be summarized by table 1.

4. The second cohomology groups of the CK algebras

4.1. The structure of the central extensions of a CK algebra

If we completely disregard the type I extensions, which are trivial for all the CK algebras, all the results obtained in section 3 can be summed up in the following theorem, which contains

Table 1. Basic and derived type II and III extension coefficients for CK algebras.

	Basic coefficients and relations	Number of them
Type IIF/IIL	$\alpha^F_{a+1\,a+2} := \alpha_{a\,a+1,a\,a+2}$ $\quad \alpha^L_{a\,a+1} := \alpha_{a\,a+2,a+1\,a+2}$ $a = 0, 1, \ldots, N-2 \quad N \geqslant 2$ $\omega_{a+3}\alpha^F_{a+1\,a+2} = \omega_{a+1}\alpha^L_{a+1\,a+2} \quad a = 0, \ldots N-3$	$2(N-1)$
Type III	$\beta_{b+1\,d+1} := \alpha_{b\,b+1,d\,d+1}$ $b = 0, 1, \ldots, N-3 \quad d = b+2, \ldots, N-1 \quad N \geqslant 3$ $\omega_b\beta = \omega_{b+1}\omega_{b+2}\beta = \omega_{b+2}\omega_{b+3}\beta = \omega_{b+4}\beta = 0$ for $\beta \equiv \beta_{b+1\,b+3}$ $\omega_b\beta = \omega_{b+2}\beta = \omega_d\beta = \omega_{d+2}\beta = 0$ for $\beta \equiv \beta_{b+1\,d+1}$ with $d = b+3, \ldots, N-1$	$(N-1)(N-2)/2$
	Derived coefficients	
Type IIF/IIL	$\alpha_{ac,a\,c+1} = \omega_{a\,c-1}\alpha^F_{c\,c+1}$ $a = 0, 1, \ldots, N-3 \quad c = a+2, \ldots, N-1 \quad N \geqslant 3$ $\alpha_{ac,a+1\,c} = \omega_{a+2\,c}\alpha^L_{a\,a+1}$ $a = 0, 1, \ldots, N-3 \quad c = a+3, \ldots, N \quad N \geqslant 3$	
Type III	$\alpha_{a\,a+2,a+1\,a+3} = -\omega_{a+2}\,\beta_{a+1\,a+3} \quad a = 0, 1, \ldots, N-3 \quad N \geqslant 3$	

the complete solution to the problem of finding the central extensions of CK algebras:

Theorem 4.1. The independent non-zero commutators of any central extension $\overline{so}_{\omega_1,\ldots,\omega_N}(N+1)$ of the CK Lie algebra $so_{\omega_1,\ldots,\omega_N}(N+1)$ can be written as

$$\begin{aligned}
&[\Omega_{ab}, \Omega_{bc}] = -\Omega_{ac}\\
&[\Omega_{ab}, \Omega_{a\,b+1}] = \omega_{ab}\Omega_{b\,b+1} + \omega_{a\,b-1}\alpha^F_{b\,b+1}\,\Xi\\
&\qquad [\Omega_{ab}, \Omega_{ac}] = \omega_{ab}\Omega_{bc} \qquad \text{for } c > b+1\\
&[\Omega_{ac}, \Omega_{a+1\,c}] = \omega_{a+1\,c}\Omega_{a\,a+1} + \omega_{a+2\,c}\alpha^L_{a\,a+1}\,\Xi\\
&\qquad [\Omega_{ac}, \Omega_{bc}] = \omega_{bc}\Omega_{ab} \qquad \text{for } b > a+1\\
&[\Omega_{a\,a+1}, \Omega_{c\,c+1}] = \beta_{a+1\,c+1}\,\Xi \qquad [\Omega_{a\,a+2}, \Omega_{a+1\,a+3}] = -\omega_{a+2}\beta_{a+1\,a+3}\,\Xi
\end{aligned} \tag{4.1}$$

where $\omega_{aa} := 1$. The extension is completely described by a number of extension coefficients.

- A single type II coefficient, α^L_{01}, which produces an extension which is non-trivial if $\omega_2 = 0$ and trivial otherwise.
- $(N-2)$ type II pairs, $\alpha^F_{12}, \alpha^L_{12}; \ldots; \alpha^F_{N-2\,N-1}, \alpha^L_{N-2\,N-1}$. Each pair of coefficients must satisfy $\omega_{a+3}\alpha^F_{a+1\,a+2} = \omega_{a+1}\alpha^L_{a+1\,a+2}$. The two extensions corresponding to the pair $\alpha^F_{a+1\,a+2}$ and $\alpha^L_{a+1\,a+2}$ are both non-trivial when $\omega_{a+1} = 0$ and $\omega_{a+3} = 0$. The two two-cocycles are simultaneously trivial otherwise.
- A single type II coefficient, $\alpha^F_{N-1\,N}$, which produces an extension which is non-trivial if $\omega_{N-1} = 0$ and trivial otherwise.
- $(N-2)$ type III extension coefficients $\beta_{13}, \beta_{24}, \ldots, \beta_{N-2\,N}$, satisfying

$$\omega\beta_{b+1\,b+3} = 0 \qquad \text{for } \omega = \omega_b, \omega_{b+1}\omega_{b+2}, \omega_{b+2}\omega_{b+3}, \omega_{b+4} \tag{4.2}$$

where when either $b = 0$ or $b = N-3$ the first or last condition, which would read $\omega_0\beta = 0$ or $\omega_{N+1}\beta = 0$ is not present. The extension corresponding to any of these non-zero coefficients is always non-trivial.

• $(N-2)(N-3)/2$ type III extension coefficients $\beta_{14}, \beta_{15}, \ldots, \beta_{1N}; \beta_{25}, \ldots, \beta_{2N}; \ldots, \ldots; \beta_{N-3\,N}$ whose indices differ by more than two. The coefficient $\beta_{b+1\,d+1}$ satisfies

$$\omega\beta_{b+1\,d+1} = 0 \qquad \text{for } \omega = \omega_b, \omega_{b+2}, \omega_d, \omega_{d+2} \tag{4.3}$$

with similar restrictions as to the actual presence of the equations involving the non-existent values ω_0 or ω_{N+1}. The extension corresponding to any of these non-zero coefficients is always non-trivial.

All type II extensions come from the pseudo-cohomology mechanism [19, 20]; if $\omega_{b+1} \neq 0 \neq \omega_{b+3}$ we obtain from (4.1)

$$\begin{aligned} [\Omega_{a\,b+1}, \Omega_{a\,b+2}] &= \omega_{a\,b+1}\left(\Omega_{b+1\,b+2} + \frac{1}{\omega_{b+1}}\alpha^F_{b+1\,b+2}\,\Xi\right) \\ [\Omega_{b+1\,c}, \Omega_{b+2\,c}] &= \omega_{b+2\,c}\left(\Omega_{b+1\,b+2} + \frac{1}{\omega_{b+3}}\alpha^L_{b+1\,b+2}\,\Xi\right) \end{aligned} \tag{4.4}$$

so we may remove the extension coefficients by means of a redefinition of the generator Ω_{b+1b+2} given by the one-cochain of non-zero coordinates $\frac{1}{\omega_{b+1}}\alpha^F_{b+1b+2} = \frac{1}{\omega_{b+3}}\alpha^L_{b+1b+2}$. When ω_{b+1} (ω_{b+3}) goes to zero the extension given by α^F_{b+1b+2}, (α^L_{b+1b+2}) might be non-trivial (because the one-cochain from it comes diverges but $\omega_{a\,b+1}/\omega_{b+1}$ does not). Due to the Jacobi identity (represented here by the equation $\omega_{b+3}\alpha^F_{b+1\,b+2} = \omega_{b+1}\alpha^L_{b+1\,b+2}$) the non-trivial extension exists if both $\omega_{b+1} = \omega_{b+3} = 0$. In terms of the standard triangular arrangement of generators, it is worth remarking that each type II coefficient $\alpha^F_{b+1\,b+2}$ appears only in the extended commutators of two horizontal neighbours in the columns of $\Omega_{0b+1}, \Omega_{0\,b+2}$, while each type II coefficient $\alpha^L_{b+1\,b+2}$ appears only in the extended commutators of two vertical neighbours in the rows of $\Omega_{b+1N}, \Omega_{b+2N}$. The corresponding extension is non-trivial only when both $\omega_{b+1} = \omega_{b+3} = 0$; this means that the algebra has two different semidirect structures (cf section 2). This is exhibited by the two rectangle boxes in the following diagram, where we have shortened $\alpha^F \equiv \alpha^F_{b+1\,b+2}$ and $\alpha^L \equiv \alpha^L_{b+1b+2}$ and we have also indicated the ω_i factors which appear in these extended commutators for the generators which are inside one of the boxes but outside the other:

$$\begin{array}{cc|ccc|cccc}
\Omega_{01}\ldots & \Omega_{0\,b} & \Omega_{0\,b+1} & \xrightarrow{\omega_{0b}\alpha^F} & \Omega_{0\,b+2} & \Omega_{0\,b+3} & \Omega_{0\,b+4} & \ldots & \Omega_{0N} \\
\ddots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
 & \Omega_{b-1\,b} & \Omega_{b-1\,b+1} & \xrightarrow{\omega_{b-1\,b}\alpha^F} & \Omega_{b-1\,b+2} & \Omega_{b-1\,b+3} & \Omega_{b-1\,b+4} & \ldots & \Omega_{b-1\,N} \\
 & & \Omega_{b\,b+1} & \xrightarrow{\alpha^F} & \Omega_{b\,b+2} & \Omega_{b\,b+3} & \Omega_{b\,b+4} & \ldots & \Omega_{b\,N} \\
\cline{3-9}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & & & \Omega_{b+1\,b+2} & \Omega_{b+1\,b+3} & \Omega_{b+1\,b+4} & \ldots & \Omega_{b+1\,N} \\
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & & & & \downarrow \alpha^L & \downarrow \omega_{b+3\,b+4}\alpha^L & & \downarrow \omega_{b+3\,N}\alpha^L \\
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & & & & \Omega_{b+2\,b+3} & \Omega_{b+2\,b+4} & \ldots & \Omega_{b+2\,N} \\
\cline{6-9}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & & & & & \Omega_{b+3\,b+4} & \ldots & \Omega_{b+3\,N} \\
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & & & & & & \ddots & \vdots \\
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & & & & & & & \Omega_{N-1\,N}.
\end{array}$$

4.2. The dimension of the second cohomology groups of the CK contracted algebras

Theorem 4.1 contains all the necessary information to determine easily the dimension of the second cohomology group $H^2(so_{\omega_1,\ldots,\omega_N}(N+1), \mathbb{R})$ of the CK algebras. This dimension is obtained as the sum of a number of completely independent contributions, each one related to the vanishing of specific groups of constants ω_i as follows.

• 1 when $\omega_2 = 0$, with two-cocycle determined by the basic coefficient α^L_{01}.

• 2 for each pair of next-neighbour zero constants, $\omega_1 = \omega_3 = 0$, $\omega_2 = \omega_4 = 0, \ldots,$ $\omega_{N-2} = \omega_N = 0$. The two-cocycles appearing with the vanishing pair $\omega_{a+1} = \omega_{a+3} = 0$ are determined by basic extension coefficients $\alpha^F_{a+1\,a+2}$ and $\alpha^L_{a+1\,a+2}$. This might amount to a subtotal of $2(N-2)$ when all pairs of second neighbours are zero, i.e. when all constants ω_i are zero.

• 1 when $\omega_{N-1} = 0$, with extension coefficient $\alpha^F_{N-1\,N}$.

• 1 for each group of zero constants $\{\omega_b, \omega_{b+2}, \omega_{b+4}\}$ or $\{\omega_b, \omega_{b+1}, \omega_{b+3}, \omega_{b+4}\}$ with extension coefficient $\beta_{b+1\,b+3}$.

• 1 for each group of zero constants $\{\omega_b, \omega_{b+2}, \omega_d, \omega_{d+2}\}$ with extension coefficient $\beta_{b+1\,d+1}$ for $d = b+3, \ldots, N-1$.

As mentioned after (4.2) and (4.3), the literal application of the two last rules may apparently involve the constants ω_0, ω_N. In these cases the corresponding conditions involving these inexistent values should be disregarded.

We can translate the previous rules into a closed formula for the dimension of the second cohomology group $H^2(so_{\omega_1\ldots\omega_N}(N+1), \mathbb{R})$. Let δ_i be defined by

$$\delta_i = \begin{cases} 1 & \omega_i = 0 \\ 0 & \omega_i \neq 0 \end{cases} \qquad (i = 1, \ldots, N) \tag{4.5}$$

then $\dim H^2(so_{\omega_1\ldots\omega_N}(N+1), \mathbb{R})$ is given in terms of the sequence $\delta_1, \delta_2, \ldots, \delta_N$ by

$$\begin{aligned}\dim(H^2(so_{\omega_1\ldots\omega_N}(N+1), \mathbb{R})) &= \delta_2 + \delta_{N-1} + 2\sum_{i=1}^{N-2} \delta_i\delta_{i+2} \\ &\quad + \sum_{i=1}^{N-2} \delta_i\delta_{i+4}[\delta_{i+2} + \delta_{i+1}\delta_{i+3} - \delta_{i+2}\delta_{i+1}\delta_{i+3}] + \sum_{i=1}^{N-3}\sum_{j=i+3}^{N} \delta_i\delta_{i+2}\delta_j\delta_{j+2}\end{aligned} \tag{4.6}$$

where $\omega_i \equiv 0$ ($\delta_i = 1$) for $i > N$. For instance, if $\omega_i = 0\ \forall\ i = 1, \ldots, N$ (flag algebra) then all $\delta_i = 1$, hence all terms in (4.6) contribute and we obtain

$$\begin{aligned}\dim(H^2(so_{0\ldots0}(N+1), \mathbb{R})) &= 2 + 2(N-2) + (N-2) + \sum_{i=1}^{N-3}(N-i-2) \\ &= 2(N-1) + \frac{(N-2)(N-1)}{2} = \frac{N(N+1)}{2} - 1.\end{aligned} \tag{4.7}$$

Each term in formula (4.6) is related to a given extension coefficient as stated in theorem 4.1 and the preceding rules.

To effectively apply the above rules, it is convenient to browse through the list of possible extension coefficients and to see whether each of them is allowed/trivial for the algebra we are dealing with or not. As a first example, consider the algebra $so_{0,\omega_2,0,0}(5)$ with $\omega_2 \neq 0$. For any CK algebra $so_{\omega_1,\omega_2,\omega_3,\omega_4}(5)$, the possible extension coefficients are: α^L_{01}, α^F_{12}, α^L_{12}, α^F_{23}, α^L_{23}, α^F_{34}; β_{13}, β_{14}, β_{24}. In this case, it is clear that the type II non-trivial ones are only α^F_{12}, α^L_{12} (as $\omega_1 = \omega_3 = 0$) and α^F_{34} (as here $\omega_{N-1} \equiv \omega_3 = 0$). Type III extension coefficient β_{13} is allowed and therefore gives a non-trivial cocycle, as $\omega_1 = 0, \omega_3 = 0$ and $\omega_4 = 0$. Type III extension β_{14} is not allowed, since ω_2 and ω_3 are not simultaneously equal to zero. Type III coefficient β_{24} is allowed (and therefore non-trivial) as $\omega_1 = 0, \omega_3 = 0$. So the dimension of the second cohomology group is equal to 5 in this case.

The dimension of H^2 for many other algebras can be derived from these rules. Although in the next section we shall give explicitly all the extended CK algebras up to $N = 4$, we mention here the result for some interesting algebras (see section 2). In some cases these cohomology groups have been known for a long time.

(1) When all ω_i are different from zero, all type III coefficients are equal to zero, and all type II (which can be different from zero) are coboundaries. Therefore, the second cohomology group of the simple pseudo-orthogonal algebra $so_{\omega_1,\ldots,\omega_N}(N+1)$ $\forall\ \omega_i \neq 0$, is trivial in accordance with the Whitehead lemma, $H^2(\mathcal{G}, \mathbb{R}) = 0$ if $\mathcal{G}$ is semisimple.

(2) If only the first constant is equal to zero, $\omega_1 = 0$, we see that the inhomogeneous algebras $iso(p,q)$ where $p + q = N$ (e.g. Euclidean and Poincaré) have no non-trivial extensions except in the case $N = 2$, where the first constant also plays the role of the last but one, and there is a single extension coefficient α_{12}^F. This result is just a rephrasing of the statement that in this case every local exponent is equivalent to zero for $N > 2$, as found by Bargmann [3] in his classical study.

(3) When $\omega_1 = \omega_2 = 0$ (all others being non-zero) the twice inhomogeneous algebras $iiso(p,q)$ have a non-trivial extension coefficient: α_{01}^L (this is just the mass for the Galilei algebra, which parametrises its second cohomology group). Generically, this is the only non-trivial extension in this family of algebras, though in the lower dimensional cases $N = 2, 3$ additional non-trivial extensions appear, as seen in the examples below.

(4) The flag algebra $ii\ldots iso(1)$ is the most contracted algebra in the CK family, and corresponds to all $\omega_i = 0$. In this case, basic type II or III extension coefficients, whenever present, lead to non-trivial extensions. Furthermore, all the conditions that these coefficients must satisfy are automatically fulfilled, as a consequence of the vanishing of all ω_i. There are $2(N-1)$ type II and $[(N-1)(N-2)/2]$ type III coefficients in this case, so that

$$H^2(so_{0,\ldots,0}(N+1), \mathbb{R}) = \mathbb{R}^{2(N-1)+[(N-1)(N-2)/2]} = \mathbb{R}^{\left[\binom{N+1}{2}-1\right]}. \tag{4.8}$$

The dimension of H^2 for the flag algebra is just equal to $\dim(so_{\omega_1,\ldots,\omega_N}(N+1)) - 1$ (see (4.7)).

To conclude this section we mention that, had we considered graded contractions from $so(N+1)$ beyond the CK family, we would have found a larger set of algebras with the $[(N+1)N/2]$-dimensional Abelian algebra as the most contracted one. Since for it all equations (3.5) are trivially satisfied and only the antisymmetry conditions (3.4) remain, the cohomology group of this Abelian algebra has dimension $\binom{N+1}{2}\left(\binom{N+1}{2}-1\right)/2$.

5. Examples: all central extensions for $N = 2, 3, 4$

We extract from the general solution in theorem 4.1 the central extensions for all the CK algebras $so_{\omega_1,\ldots,\omega_N}(N+1)$ for $N = 2, 3, 4$ [15]. We remark that our results cover in a single stroke a large family of Lie algebras; in particular, the family $so_{\omega_1,\omega_2,\omega_3,\omega_4}(4+1)$ contains all relativistic and non-relativistic $3+1$ kinematical algebras, the cohomology of which can be then read off directly.

5.1. $\overline{so}_{\omega_1,\omega_2}(3)$

There are two central extension coefficients of type II:

$$\alpha_{01}^L \equiv \alpha_{02,12} \qquad \alpha_{12}^F \equiv \alpha_{01,02} \tag{5.1}$$

which are not constrained by any additional condition. The Lie brackets of $\overline{so}_{\omega_1,\omega_2}(3)$ are

$$[\Omega_{01}, \Omega_{02}] = \omega_1\Omega_{12} + \alpha_{12}^F\,\Xi \qquad [\Omega_{01}, \Omega_{12}] = -\Omega_{02} \qquad [\Omega_{02}, \Omega_{12}] = \omega_2\Omega_{01} + \alpha_{01}^L\,\Xi. \tag{5.2}$$

The triviality of any such extension is governed by the second and last-but-one constants in the list ω_i. In this case, these are the second and the first. Thus α_{01}^L is trivial if $\omega_2 \neq 0$, and α_{12}^F is trivial if $\omega_1 \neq 0$. This is exhibited by the redefinitions

$$\Omega_{01} \to \Omega_{01} + \frac{\alpha_{01}^L}{\omega_2}\Xi \qquad \omega_2 \neq 0 \qquad \Omega_{12} \to \Omega_{12} + \frac{\alpha_{12}^F}{\omega_1}\Xi \qquad \omega_1 \neq 0. \tag{5.3}$$

Thus, $\dim[H^2(so_{\omega_1,\omega_2}(3), \mathbb{R})]$ is equal to:

- 0 for the simple Lie algebras $so(3)$ and $so(2,1)$ (both ω_1 and $\omega_2 \neq 0$),
- 1 for the two-dimensional Euclidean algebra, which appears for the constants $(0,1)$ (extension coefficient α_{12}^F) and $(1,0)$ (extension coefficient α_{01}^L),
- 1 for the $(1+1)$-Poincaré algebra, which also appears twice, as $(0,-1)$ and $(-1,0)$, respectively with extension coefficients α_{12}^F and α_{01}^L,
- 2 for the $(1+1)$-Galilei algebra, which appear for constants $(0,0)$, with both extension coefficients α_{12}^F and α_{01}^L. Physically, these extensions are parametrized by the mass and a constant force (see [7] and references therein and [21]).

5.2. $\overline{so}_{\omega_1,\omega_2,\omega_3}(4)$

The full set of extension possibilities appears first in this case. However, there are some non-generic coincidences. There are four basic extension coefficients of type II, and one of type III:

$$\alpha_{01}^L \equiv \alpha_{02,12} \qquad \alpha_{12}^F \equiv \alpha_{01,02} \qquad \alpha_{12}^L \equiv \alpha_{13,23} \qquad \alpha_{23}^F \equiv \alpha_{12,13} \qquad \beta_{13} \equiv \alpha_{01,23} \tag{5.4}$$

which must satisfy

$$\omega_3\alpha_{12}^F = \omega_1\alpha_{12}^L \qquad \omega_1\omega_2\beta_{13} = 0 \qquad \omega_2\omega_3\beta_{13} = 0. \tag{5.5}$$

Then, equations (4.1) give the commutation rules of $\overline{so}_{\omega_1,\omega_2,\omega_3}(4)$:

$$\begin{aligned}
&[\Omega_{01},\Omega_{02}] = \omega_1\Omega_{12} + \alpha_{12}^F\Xi \qquad [\Omega_{01},\Omega_{12}] = -\Omega_{02} \qquad [\Omega_{02},\Omega_{12}] = \omega_2\Omega_{01} + \alpha_{01}^L\Xi\\
&[\Omega_{01},\Omega_{03}] = \omega_1\Omega_{13} \qquad [\Omega_{01},\Omega_{13}] = -\Omega_{03} \qquad [\Omega_{03},\Omega_{13}] = \omega_3(\omega_2\Omega_{01} + \alpha_{01}^L\Xi)\\
&[\Omega_{02},\Omega_{03}] = \omega_1(\omega_2\Omega_{23} + \alpha_{23}^F\Xi) \qquad [\Omega_{02},\Omega_{23}] = -\Omega_{03} \qquad [\Omega_{03},\Omega_{23}] = \omega_3\Omega_{02}\\
&[\Omega_{12},\Omega_{13}] = \omega_2\Omega_{23} + \alpha_{23}^F\Xi \qquad [\Omega_{12},\Omega_{23}] = -\Omega_{13} \qquad [\Omega_{13},\Omega_{23}] = \omega_3\Omega_{12} + \alpha_{12}^L\Xi\\
&[\Omega_{01},\Omega_{23}] = \beta_{13}\Xi \qquad [\Omega_{02},\Omega_{13}] = -\omega_2\beta_{13}\Xi \qquad [\Omega_{03},\Omega_{12}] = 0.
\end{aligned} \tag{5.6}$$

The extension coefficient α_{01}^L produces a non-trivial cocycle when the second constant $\omega_2 = 0$, the extension α_{23}^F is non-trivial when the last-but-one constant (ω_2 again in this case) is zero and the extensions given by α_{12}^F and α_{12}^L are non-trivial when $\omega_1 = \omega_3 = 0$. The extension determined by β_{13} is only present (see (5.5)) when $\omega_2 = 0$ or when $\omega_1 = \omega_3 = 0$, and whenever it appears, it is non-trivial. The redefinition of generators displaying the triviality of type II extensions is:

$$\begin{aligned}
&\Omega_{01} \to \Omega_{01} + \frac{\alpha_{01}^L}{\omega_2}\Xi \qquad \text{if } \omega_2 \neq 0\\
&\Omega_{12} \to \Omega_{12} + \frac{\alpha_{12}^L}{\omega_3}\Xi \qquad \text{if } \omega_3 \neq 0\\
&\Omega_{12} \to \Omega_{12} + \frac{\alpha_{12}^F}{\omega_1}\Xi \qquad \text{if } \omega_1 \neq 0\\
&\Omega_{23} \to \Omega_{23} + \frac{\alpha_{23}^F}{\omega_2}\Xi \qquad \text{if } \omega_2 \neq 0.
\end{aligned} \tag{5.7}$$

Note that when ω_1 and ω_3 both differ from zero, equation (5.5) guarantees that both expressions for the redefinition of Ω_{12} indeed coincide.

To conclude the analysis we now give $\dim[H^2(so_{\omega_1,\omega_2,\omega_3}(4),\mathbb{R})]$:

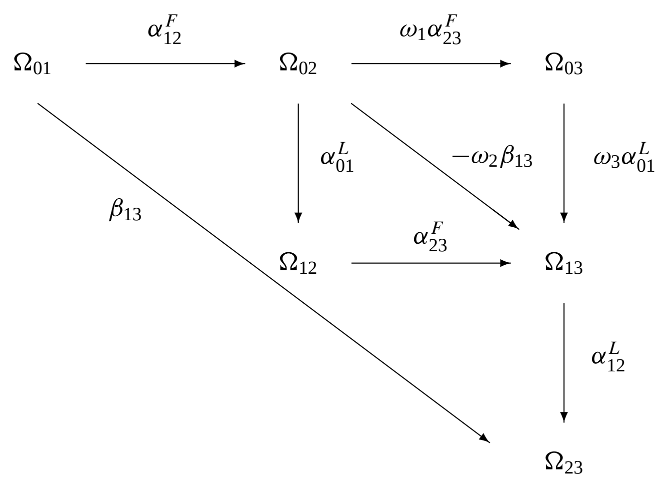
• 3 for $\omega_2 = 0$ with either ω_1 or ω_3 non-zero: non-trivial extension coefficients α^L_{01}, α^F_{23} and β_{13}. Examples here are both $(2+1)$ Newton–Hooke algebras $(\pm 1, 0, 1)$, $(1, 0, \pm 1)$ and the $(2+1)$ Galilean one $iiso(2)$ $(0, 0, 1)$, $(1, 0, 0)$. Also $iiso(1, 1)$ $(-1, 0, 0)$, $(0, 0, -1)$ and $t_4 \odot (so(1, 1) \oplus so(1, 1))$ $(-1, 0, -1)$.

• 3 for $\omega_1 = \omega_3 = 0$ and $\omega_2 \neq 0$; non-trivial extensions are α^F_{12}, α^L_{12} and β_{13}. Here we find the $(2+1)$ Carroll algebra $(0, 1, 0)$ and $ii'so(1, 1)$ $(0, -1, 0)$.

• 5 for the most contracted algebra in the CK family with $\omega_1 = \omega_2 = \omega_3 = 0$; it corresponds to the flag space algebra $iiiso(1)$.

• 0 for all the remaining algebras. Up to isomorphisms these include the semisimple ones $so(4)$ (once), $so(3, 1)$ (four times), $so(2, 2)$ (three times), the three-dimensional Euclidean $iso(3)$ (two times) and Poincaré algebras $iso(2, 1)$ (six times).

For convenience we include below the standard triangular diagram with all the extended commutators for $\overline{so}_{\omega_1,\omega_2,\omega_3}(4)$. An arrow between generators A and B means that a central Ξ-term, with coefficient indicated near the arrow, is added to the non-extended commutator $[A, B]$. In the usual kinematical interpretation, the generators may be translated as $\Omega_{01} \to H$ (Hamiltonian), $\Omega_{02} \to P_1$, $\Omega_{03} \to P_2$ (momenta), $\Omega_{12} \to K_1$, $\Omega_{13} \to K_2$ (boosts), $\Omega_{23} \to J$ (rotation).



We recall that up to now we have referred in this paper to the cohomology groups of Lie *algebras* and not of groups. As mentioned they do not necessarily coincide [3, 8, 9], as illustrated by the standard example of the $G(2+1)$ Galilei group (our $SO_{(0,0,1)}(4)$) for which $\dim[H^2(G(2+1), U(1))] = 2$ although $\dim[H^2(\mathcal{G}(2+1), \mathbb{R})] = 3$, a fact known [7] (see also [22] for a recent discussion) already for 25 years†. With $\Omega_{01} \equiv H$ and space rotation generator $\Omega_{23} \equiv J$, we see that the algebra commutator $[H, J]$ admits an extension through β_{13} but that the compactness condition on the space rotation generator J, relevant for the Galilei group, forces the coefficient β_{13} to disappear. This is because under a rotation generated by J in the extended algebra, H transforms by

† This well known result has attracted a renewed interest [23], specially in relation with the absence of non-relativistic planar systems with exotic angular momentum (anyons) [24].

$H \to \exp(\theta J) H \exp(-\theta J) = H - \theta\beta_{13}\Xi$. Since J is compact the rotations $\theta = 2\pi$ (with $\omega_3 = 1$) and $\theta = 0$ should coincide, which forces $\beta_{13} = 0$ and reduces in one dimension the group cohomology. In general, within the CK family of groups as obtained by exponentiation of the matrix representation (2.5) of the CK Lie algebra, the one-parameter subgroup generated by Ω_{ab} is compact if $\omega_{ab} > 0$ and non-compact otherwise. This implies that the extension coefficient $\beta_{a+1\,c+1}$, which appears in the extended commutator $[\Omega_{a\,a+1}, \Omega_{c\,c+1}] = \beta_{a+1\,c+1}\Xi$ for the algebra, does not correspond to a group extension whenever at least one of the generators $\Omega_{a\,a+1}$ or $\Omega_{c\,c+1}$ corresponds to a compact one-parameter subgroup, that is when either $\omega_{a+1} > 0$ or $\omega_{c+1} > 0$.

5.3. $\overline{so}_{\omega_1,\omega_2,\omega_3,\omega_4}(5)$

There are six basic extension coefficients of type II and three of type III:

$$\begin{array}{lll} \alpha^L_{01} \equiv \alpha_{02,12} & \alpha^L_{12} \equiv \alpha_{13,23} & \alpha^L_{23} \equiv \alpha_{24,34} \\ \alpha^F_{12} \equiv \alpha_{01,02} & \alpha^F_{23} \equiv \alpha_{12,13} & \alpha^F_{34} \equiv \alpha_{23,24} \\ \beta_{13} \equiv \alpha_{01,23} & \beta_{14} \equiv \alpha_{01,34} & \beta_{24} \equiv \alpha_{12,34} \end{array} \tag{5.8}$$

verifying the additional conditions:

$$\begin{array}{lll} \omega_3\alpha^F_{12} = \omega_1\alpha^L_{12} & \omega_4\alpha^F_{23} = \omega_2\alpha^L_{23} & \\ \omega_1\omega_2\beta_{13} = 0 & \omega_2\omega_3\beta_{13} = 0 & \omega_4\beta_{13} = 0 \\ \omega_2\beta_{14} = 0 & \omega_3\beta_{14} = 0 & \\ \omega_1\beta_{24} = 0 & \omega_2\omega_3\beta_{24} = 0 & \omega_3\omega_4\beta_{24} = 0. \end{array} \tag{5.9}$$

Therefore the Lie brackets of the extended CK algebras $\overline{so}_{\omega_1,\omega_2,\omega_3,\omega_4}(5)$ are

$$\begin{array}{ll} [\Omega_{01}, \Omega_{02}] = \omega_1\Omega_{12} + \alpha^F_{12}\Xi & [\Omega_{02}, \Omega_{12}] = \omega_2\Omega_{01} + \alpha^L_{01}\Xi \\ [\Omega_{12}, \Omega_{13}] = \omega_2\Omega_{23} + \alpha^F_{23}\Xi & [\Omega_{13}, \Omega_{23}] = \omega_3\Omega_{12} + \alpha^L_{12}\Xi \\ [\Omega_{23}, \Omega_{24}] = \omega_3\Omega_{34} + \alpha^F_{34}\Xi & [\Omega_{24}, \Omega_{34}] = \omega_4\Omega_{23} + \alpha^L_{23}\Xi \\ [\Omega_{13}, \Omega_{14}] = \omega_2(\omega_3\Omega_{34} + \alpha^F_{34}\Xi) & [\Omega_{04}, \Omega_{14}] = \omega_3\omega_4(\omega_2\Omega_{01} + \alpha^L_{01}\Xi) \\ [\Omega_{02}, \Omega_{03}] = \omega_1(\omega_2\Omega_{23} + \alpha^F_{23}\Xi) & [\Omega_{14}, \Omega_{24}] = \omega_4(\omega_3\Omega_{12} + \alpha^L_{12}\Xi) \\ [\Omega_{03}, \Omega_{04}] = \omega_1\omega_2(\omega_3\Omega_{34} + \alpha^F_{34}\Xi) & [\Omega_{03}, \Omega_{13}] = \omega_3(\omega_2\Omega_{01} + \alpha^L_{01}\Xi) \end{array}$$
$$\begin{array}{lll} [\Omega_{01}, \Omega_{23}] = \beta_{13}\Xi & [\Omega_{02}, \Omega_{13}] = -\omega_2\beta_{13}\Xi & [\Omega_{01}, \Omega_{34}] = \beta_{14}\Xi \\ [\Omega_{12}, \Omega_{34}] = \beta_{24}\Xi & [\Omega_{13}, \Omega_{24}] = -\omega_3\beta_{24}\Xi & \end{array} \tag{5.10}$$

the remaining commutators being as in the non-extended case (2.4).

We display the explicit result for each CK algebra $so_{\omega_1,\omega_2,\omega_3,\omega_4}(5)$ in table 2. The first column shows the number of simple contractions (the number of coefficients ω_a set equal to zero). The second schematically names the centrally extended Lie algebras. The third specifies the coefficients ω_a different from zero together with the non-trivial central-extension coefficients allowed. Finally, the fourth gives $\dim[H^2(so_{\omega_1,\omega_2,\omega_3,\omega_4}(5), \mathbb{R})]$ as a sum of the type II and type III contributions. Note that the only kinematical algebras in $(3+1)$ dimensions which have non-trivial central extensions (and hence projective representations) are the $(3+1)$ oscillating Newton–Hooke $(1, 0, 1, 1)$, expanding Newton–Hooke $(-1, 0, 1, 1)$ and Galilean $(0, 0, 1, 1)$ algebras, all of them of 'absolute time' [16]. This table can be used as an example of how to compute $\dim[H^2(\mathcal{G}, \mathbb{R})]$ from theorem 4.1.

Table 2. Non-trivial central extensions $\overline{so}_{\omega_1,\omega_2,\omega_3,\omega_4}(5)$ of $so_{\omega_1,\omega_2,\omega_3,\omega_4}(5)$. The constants ω_i appearing explicitly are assumed to be different from zero.

#	Extended algebra	(CK constants)	(Non-trivial ext. coefficients)	dim H^2
0	$\overline{so}(5)$	$(\omega_1, \omega_2, \omega_3, \omega_4)$		0
	$\overline{so}(4,1)$			
	$\overline{so}(3,2)$			
1	$\overline{iso}(4)$	$(0, \omega_2, \omega_3, \omega_4)$ or $(\omega_1, \omega_2, \omega_3, 0)$		0
	$\overline{iso}(3,1)$			
	$\overline{iso}(2,2)$			
	$\bar{t}_6(so(3) \oplus so(2))$	$(\omega_1, 0, \omega_3, \omega_4)$	$[\alpha^L_{01}]$ or	$1+0$
	$\bar{t}_6(so(3) \oplus so(1,1))$	$(\omega_1, \omega_2, 0, \omega_4)$	$[\alpha^F_{34}]$	
	$\bar{t}_6(so(2,1) \oplus so(2))$			
	$\bar{t}_6(so(2,1) \oplus so(1,1))$			
2	$\overline{iiso}(3)$	$(0, 0, \omega_3, \omega_4)$	$[\alpha^L_{01}]$ or	$1+0$
	$\overline{iiso}(2,1)$	$(\omega_1, \omega_2, 0, 0)$	$[\alpha^F_{34}]$	
	$\overline{ii'so}(3)$	$(0, \omega_2, \omega_3, 0)$		0
	$\overline{ii'so}(2,1)$			
	$\overline{it}_4(so(2) \oplus so(2))$	$(0, \omega_2, 0, \omega_4)$	$[\alpha^L_{12}, \alpha^F_{12}, \alpha^F_{34}; \beta_{24}]$ or	$3+1$
	$\overline{it}_4(so(2) \oplus so(1,1))$	$(\omega_1, 0, \omega_3, 0)$	$[\alpha^L_{01}, \alpha^L_{23}, \alpha^F_{23}; \beta_{13}]$	
	$\overline{it}_4(so(1,1) \oplus so(1,1))$			
	$\bar{t}_6(iso(2) \oplus so(2))$	$(\omega_1, 0, 0, \omega_4)$	$[\alpha^L_{01}, \alpha^F_{34}; \beta_{14}]$	$2+1$
	$\bar{t}_6(iso(2) \oplus so(1,1))$			
	$\bar{t}_6(iso(1,1) \oplus so(1,1))$			
3	$\overline{iiiso}(2)$	$(0, 0, 0, \omega_4)$	$[\alpha^L_{01}, \alpha^F_{12}, \alpha^L_{12}, \alpha^F_{34}; \beta_{14}, \beta_{24}]$ or	$4+2$
	$\overline{iiiso}(1,1)$	$(\omega_1, 0, 0, 0)$	$[\alpha^L_{01}, \alpha^F_{23}, \alpha^L_{23}, \alpha^F_{34}, \beta_{13}, \beta_{14}]$	
	$\overline{iii'so}(2)$	$(0, 0, \omega_3, 0)$	$[\alpha^L_{01}, \alpha^F_{23}, \alpha^L_{23}; \beta_{13}, \beta_{24}]$ or	$3+2$
	$\overline{iii'so}(1,1)$	$(0, \omega_2, 0, 0)$	$[\alpha^F_{12}, \alpha^L_{12}, \alpha^F_{34}; \beta_{13}, \beta_{24}]$	
4	$\overline{iiiiso}(1)$	$(0, 0, 0, 0)$	$[\alpha^L_{01}, \alpha^F_{12}, \alpha^L_{12}, \alpha^F_{23}, \alpha^L_{23}, \alpha^F_{34}; \beta_{13}, \beta_{14}, \beta_{24}]$	$6+3$

6. Concluding remarks

We have characterized with generality the second cohomology groups $H^2(so_{\omega_1\ldots\omega_N}(N+1), \mathbb{R})$ of the CK family of algebras $so_{\omega_1\ldots\omega_N}(N+1)$, which is a particular subfamily of all graded contractions of the $so(N+1)$ algebra. The algebras in the CK family can be described in a simultaneous and economical way using N real 'contraction' coefficients $\omega_1, \omega_2, \ldots, \omega_N$. The procedure also exhibits the origin of the various central extensions and in particular differentiates clearly those which come from contractions of trivial extensions from those which do not.

It is well known that, by Whitehead's lemma, all semisimple Lie algebras have trivial second cohomology groups and that by the Levi–Mal'čev theorem any finite-dimensional Lie algebra $\mathcal{G}$ is the semidirect extension of a semisimple algebra and the radical of $\mathcal{G}$. Since inhomogeneous algebras come from contraction, our procedure may be applied to find the cohomology groups of other inhomogeneous algebras as well; in particular, one could start from the real simple algebras of the A_l and C_l series. There are several CK families of algebras (see [25] for a cursory description) and *any* simple real Lie algebra appears as a member of some family. We have discussed here only the orthogonal CK family, which include the simple algebras $so(N+1)$ and $so(p,q)$ in the B_l and D_l series, as well as their (quasi-simple) contractions. A similar approach would lead to a complete characterization of the second cohomology groups for quasi-simple algebras of inhomogeneous type obtained by contraction from other real simple Lie algebras. This will be matter for further work.

Another possible application of the contraction method is the search for Casimir operators of inhomogeneous algebras. The number of primitive Casimirs of a simple algebra $\mathcal{G}$ is equal to its rank l, which in turn is equal to the different primitive invariant polynomials which can be constructed on $\mathcal{G}$. Thus, the graded contraction approach allows us, in principle, to find central elements of the enveloping algebras by contracting the original l Casimir–Racah operators. Clearly, the procedure does not permit us to find *all* the Casimirs of an *arbitrary* contraction of a simple Lie algebra of rank l, since the final step is always an Abelian algebra (hence with as many primitive Casimirs as generators) and $\dim\mathcal{G} > l$. However, within the CK family the number of functionally independent Casimirs remains constant (see [26]). This provides another justification for the name 'quasi-simple' given to its members, and explains in a simple way why e.g. the number of Casimir operators for the simple de Sitter algebra and the non-simple Poincaré one is the same.

The same kind of approach we have pursued here for studying the second cohomology groups of the CK algebras has been developed to study their deformations (in the sense of [27–29]). In particular, a whole family of deformations of inhomogeneous Lie algebras [30], or working to first order, of the corresponding bialgebras [31], has been found. The semidirect structure of the 'classical' CK $\omega_1 = 0$ inhomogeneous Lie algebras becomes [32] a bicrossproduct [33] structure for their CK deformed counterparts. Whether or not this extends to the deformations of other semidirect structures associated with the vanishing of any ω_a requires further study. A related problem would be the analysis of the structure of the deformation of inhomogeneous Lie algebras from the present graded-contraction point of view, for which central extensions should appear as cocycle-bicrossproducts. These questions are worth studying.

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